2004 COMP.DSP Conference; Cannon Falls, MN, July 29-30, 2004

Frequency Sampling Filters: The Lost Art

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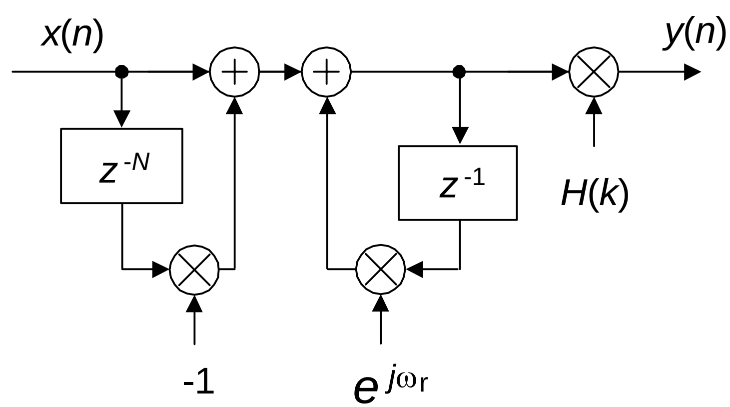


Frequency Sampling Filters

► ***Frequency sampling filters*** **are used to implement linear-phase FIR filters.**

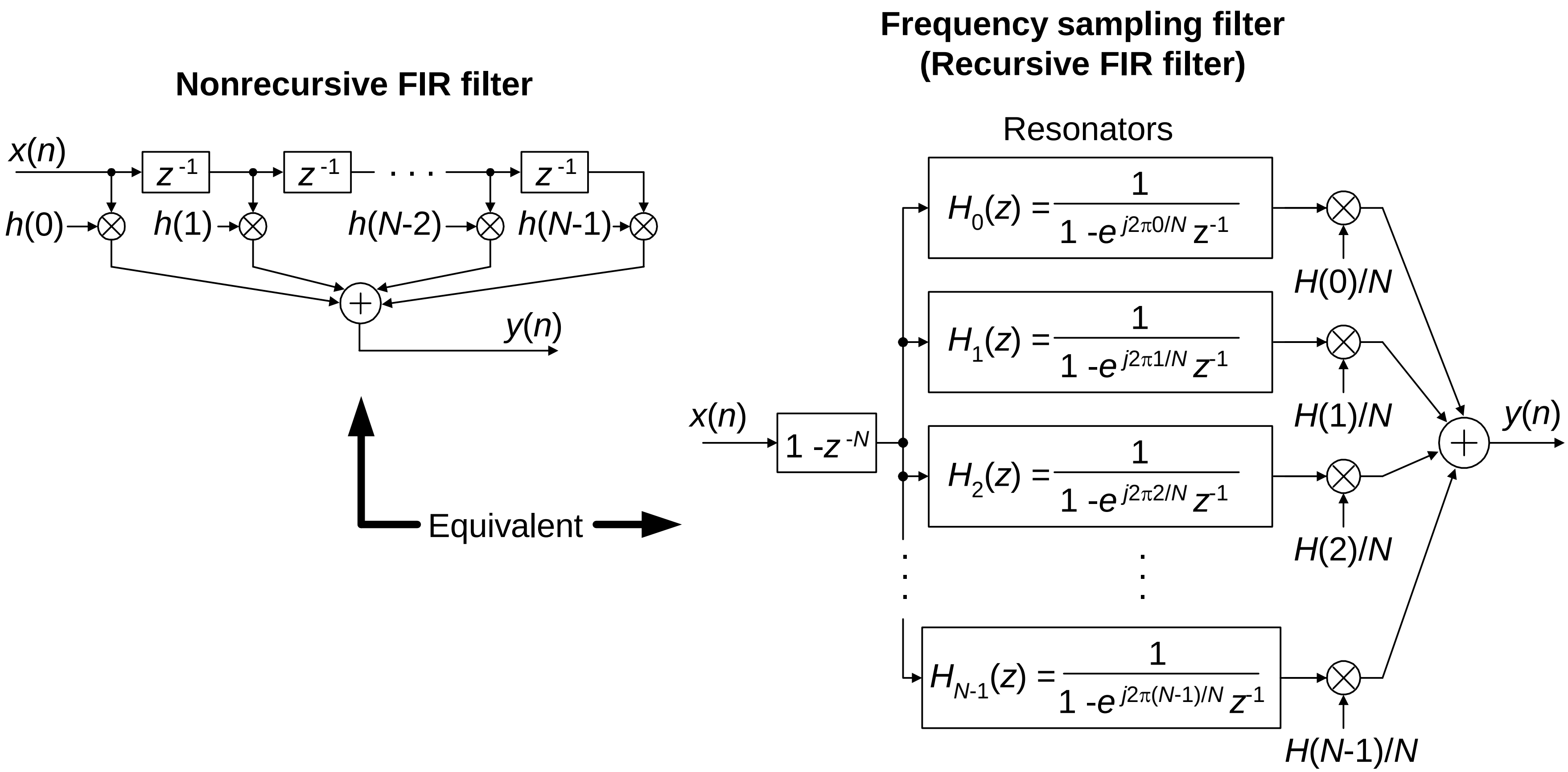
► **Can be more efficient than Parks-McClellan-designed filters for certain applications,**

  - when passband ≤ f_s/4. (f_s is the sampling rate.)

► **Rarely covered in DSP textbooks or classrooms.**

► **Here we'll spread the gospel of frequency sampling filters (FSFs).**

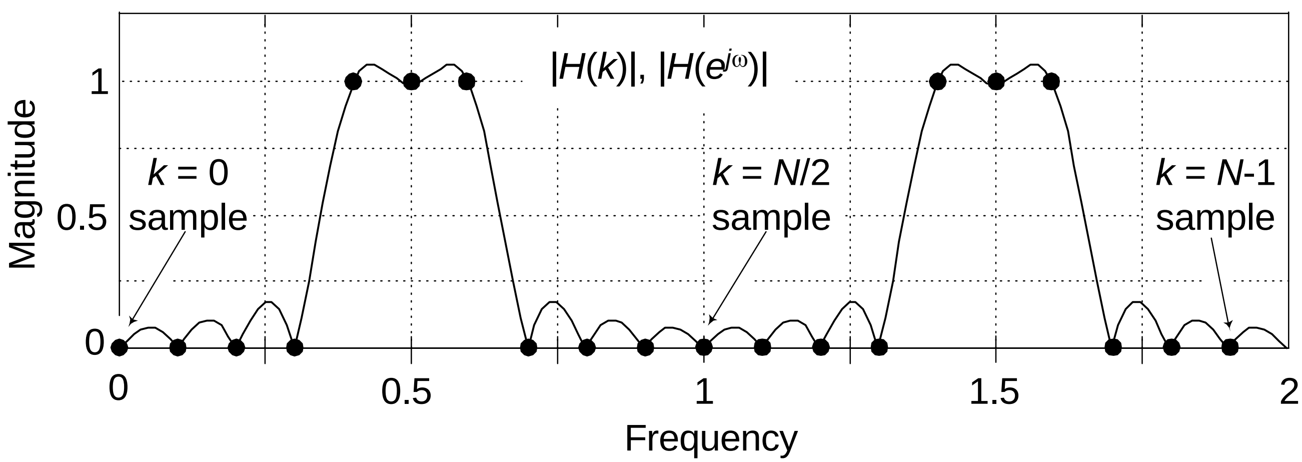
- **Frequency sampling filters were founded upon the equivalency of:**
  - **a traditional *N*-tap nonrecursive (direct convolution) FIR filter, and**
  - **a comb filter in cascade with a bank of *N* complex resonators.**
- **Coefficients *h*(*k*) and *H*(*k*) gain factors assumed to be complex.**
- ***H*(*k*) gain factors are the discrete Fourier transform of the *h*(*k*).**



- **Basis of FSF design is the definition of a desired FIR filter frequency response**
  - **in the form of *H*(*k*) frequency-domain samples,**
  - **whose magnitudes are depicted as dots in the figure below.**

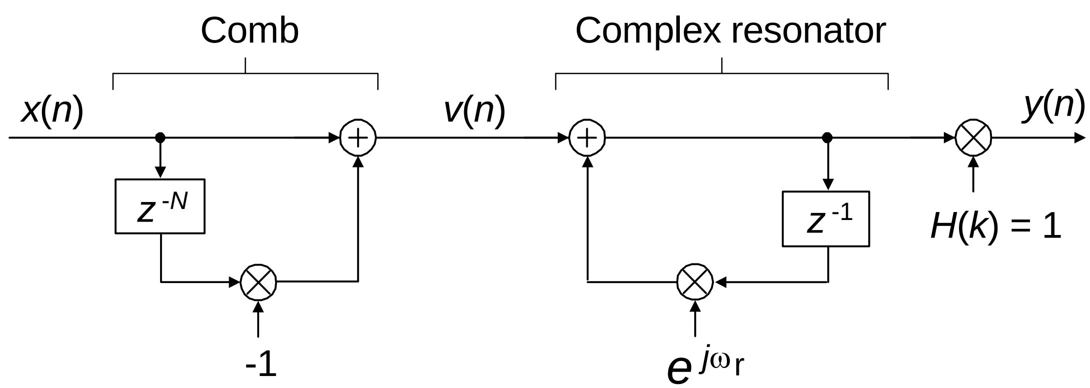
- ***H*(*k*) sample values are used as gain factors**
  - **following the resonators in the FSF structure.**

- **Frequency axis measured in π radians,**
  - **2π radians f_s (sample rate in Hz).**



- **FSFs are efficient because most *H*(*k*) are zero-valued,**
  - **corresponding to the stopband, and**
  - **need not be implemented.**

- **Simplest FSF is a single section of a complex FSF:**
  - **a comb filter followed by a single complex digital resonator.**



- **The 1/*N* gain factor following a resonator in a standard FSF is omitted, for simplicity.**
  - **The 1/*N* scaling factor will be discussed later.**

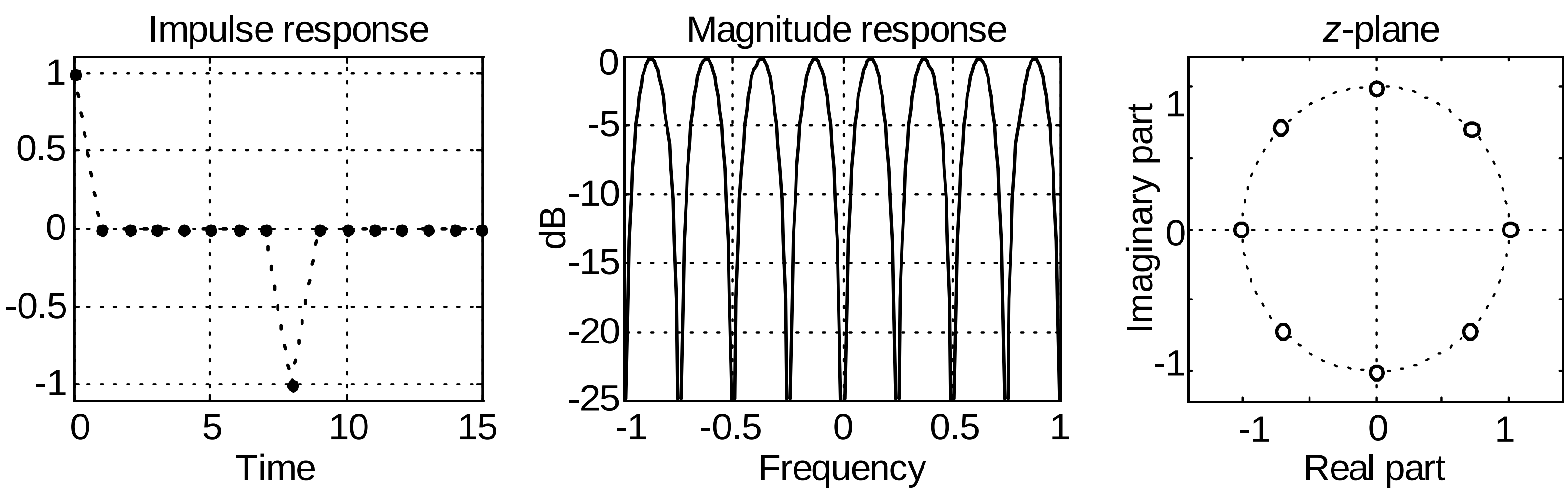
► **First, consider nonrecursive comb filter whose time-domain difference equation is**

$$v(n) = x(n) - x(n-N)$$

- **z-domain transfer function and frequency response are:**

$$H_{comb}(z) = \frac{V(z)}{X(z)} = 1 - z^{-N}, \quad and \quad H_{comb}(e^{j\omega}) = e^{-j(\omega N-\pi)/2}\, 2\sin(\omega N/2).$$

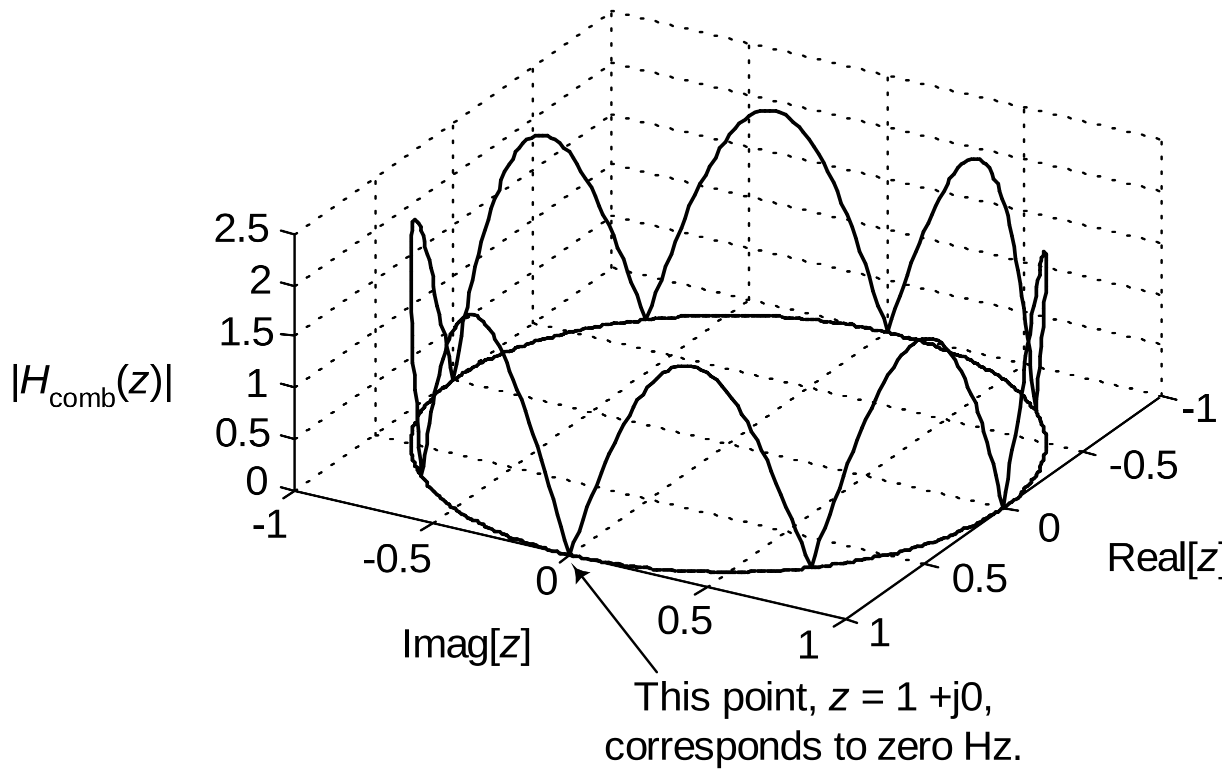
► **For N = 8, behavior is as shown below.**



(a) (b) (c)

► $H_{comb}(z)$ **has** N **periodically-spaced zeros around** z**-plane's unit circle,**

- **located at** $z(k) = e^{j2\pi k/N}$**, where** $k = 0, 1, 2, ..., N-1$**.**



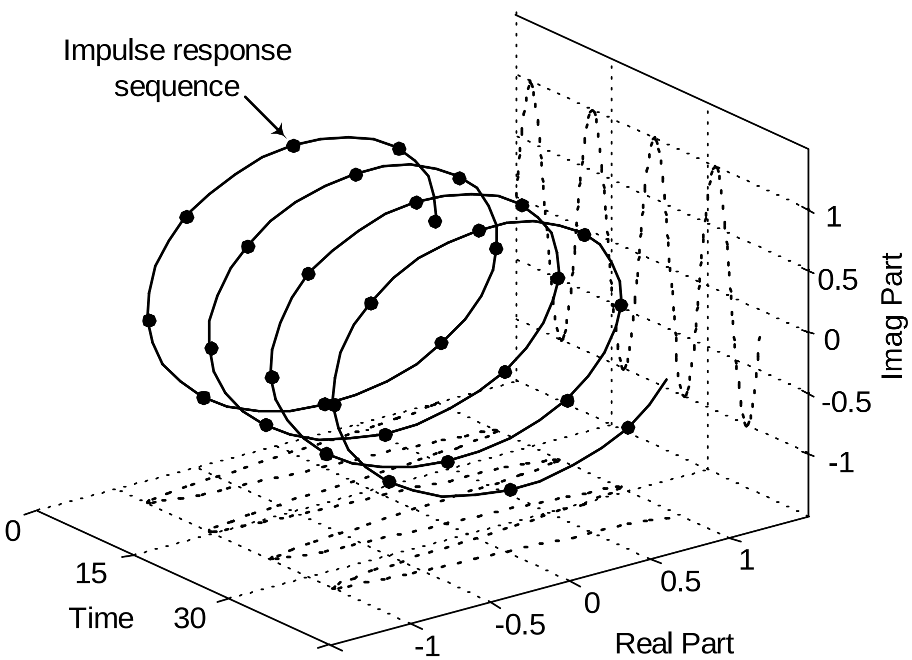
► **Now, for the complex resonator:**

- **Its time-domain difference equation and transfer function are:**

$$y(n) = v(n) + e^{j\omega_r} y(n\text{-}1), \quad \text{and} \quad H_{res}(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 - e^{j\omega_r} z^{-1}}$$

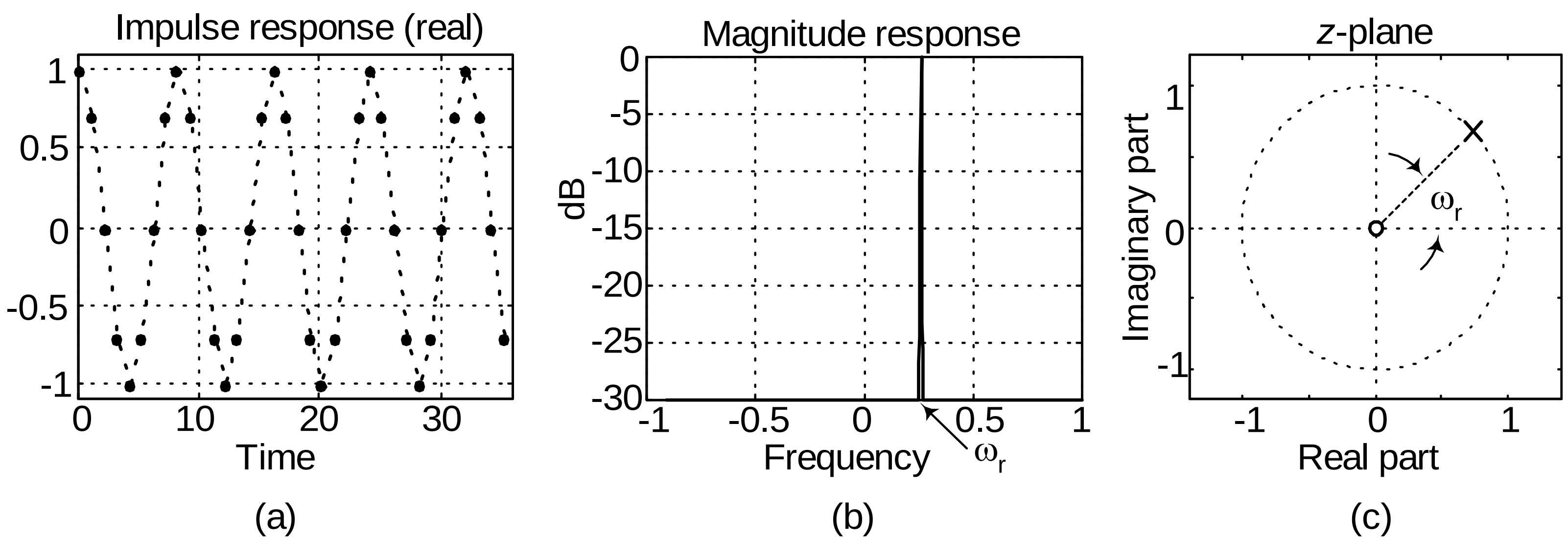
- **where the angle ω_r, $-\pi \le \omega_r \le \pi$, determines the resonant frequency.**

► **Resonator's impulse response, for $\omega_r = \pi/4$, is:**



► **Impulse response is a complex sinusoid.**

► **$H_{res}(z)$ has a pole at $z = e^{j\omega_r}$, on the unit circle at an angle of $\omega_r = \pi/4$.**



Transfer function of single-section complex FSF is

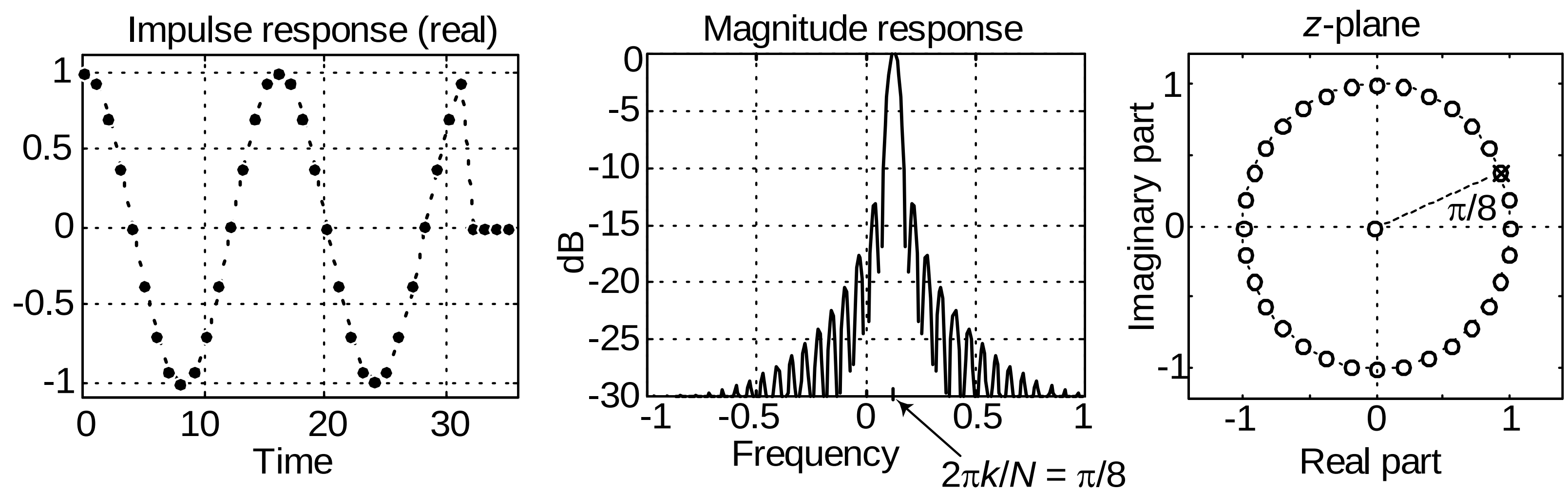
$$H(z) = H_{comb}(z)\, H_{res}(z)\, H(k) = (1 - z^{-N}) \frac{H(k)}{1 - e^{j\omega_r} z^{-1}}.$$

► **Restrict resonator's resonant frequency ω_r to be $2\pi k/N$,**

- **resonator's pole located atop one of the comb's zeros, yielding**
- **FSF transfer function of**

$$H_{ss}(z) = (1 - z^{-N}) \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

► **For $N = 32$, $k = 2$, and $H(2) = 1$, FSF behavior is:**



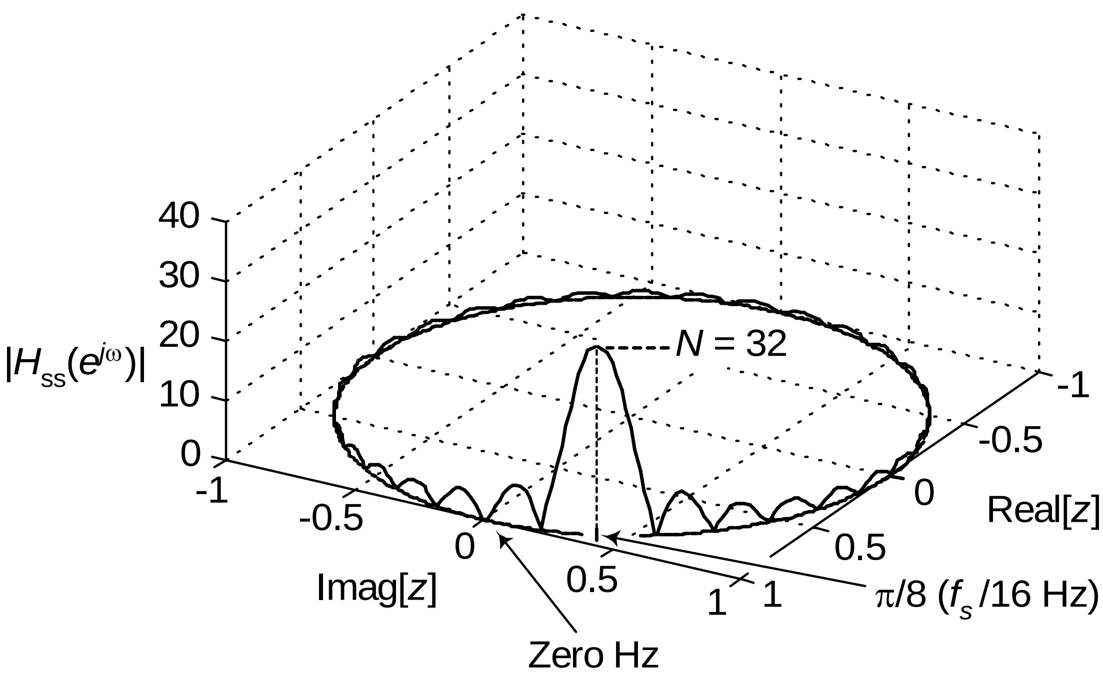
(a) (b) (c)

► **Impulse response is a truncated complex sinusoid.**

- Frequency magnitude response is a sin(x)/x-like function.
- Pole/zero cancelation at $2\pi k/N = \pi/8$ radians ($f_s/16$ Hz).
- Frequency response is:

$$H_{ss}(e^{j\omega}) = H_{ss}(z)|_{z=e^{j\omega}} = e^{-j\omega(N-1)/2}\, e^{-j\pi k/N}\, H(k) \frac{\sin(\omega N/2)}{\sin(\omega/2 - \pi k/N)}.$$

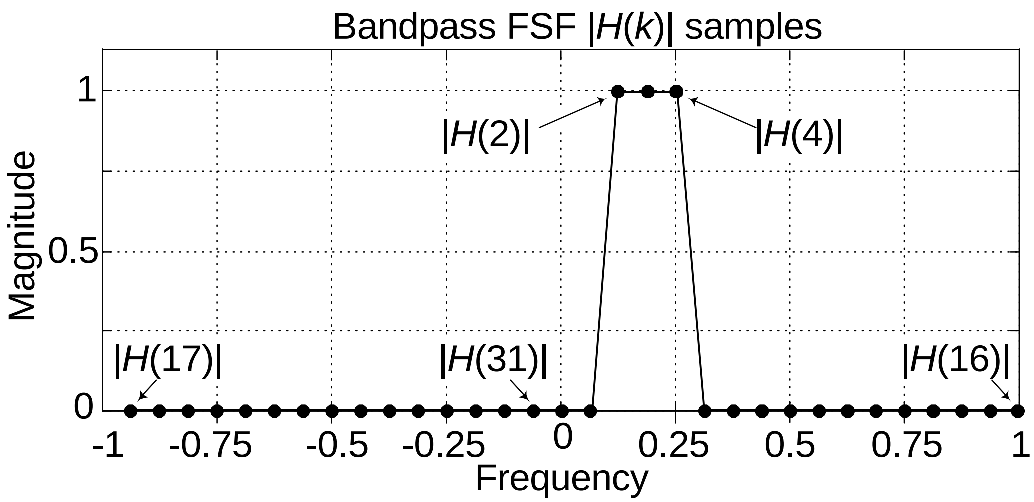
- $H_{ss}(e^{j\omega})$ has linear phase.
- Max magnitude response of $H_{ss}(e^{j\omega})$ is N when $|H(k)| = 1$.

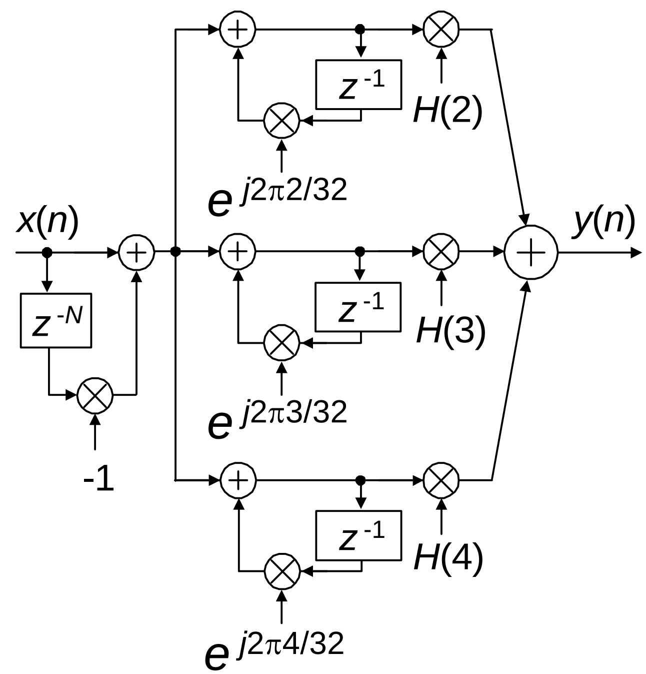


Multisection Complex FSFs

► **Practical FSFs use multiple resonator sections,**

- **like the following bandpass filter.**
- **Here's a 3-section complex FSF: $N = 32$, $H(2) = H(3) = H(4) = 1$.**





► **Transfer function of the 3-section, $N = 32$, complex FSF is**

$$H_{\text{cplx},3}(z) = (1 - z^{-32})\left[\frac{H(2)}{1 - e^{j2\pi 2/32}z^{-1}} + \frac{H(3)}{1 - e^{j2\pi 3/32}z^{-1}} + \frac{H(4)}{1 - e^{j2\pi 4/32}z^{-1}}\right] = (1 - z^{-32})\sum_{k=2}^{4}\frac{H(k)}{1 - e^{j2\pi k/32}z^{-1}}$$

- ► Transfer function of a *general* complex FSF is

$$H_{\text{cplx}}(z) = (1 - z^{-N}) \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}}.$$

- ► For an $N = 32$ complex FSF we could have up to 32 resonators,
  - but in practice only a few resonators are needed for narrowband filters.

- ► Frequency response of a multisection complex FSF is

$$H_{\text{cplx}}(e^{j\omega}) = e^{-j\omega(N-1)/2} \sum_{k=0}^{N-1} \frac{H(k)\, e^{-j\pi k/N} \sin(\omega N/2)}{\sin(\omega/2 - \pi k/N)}.$$

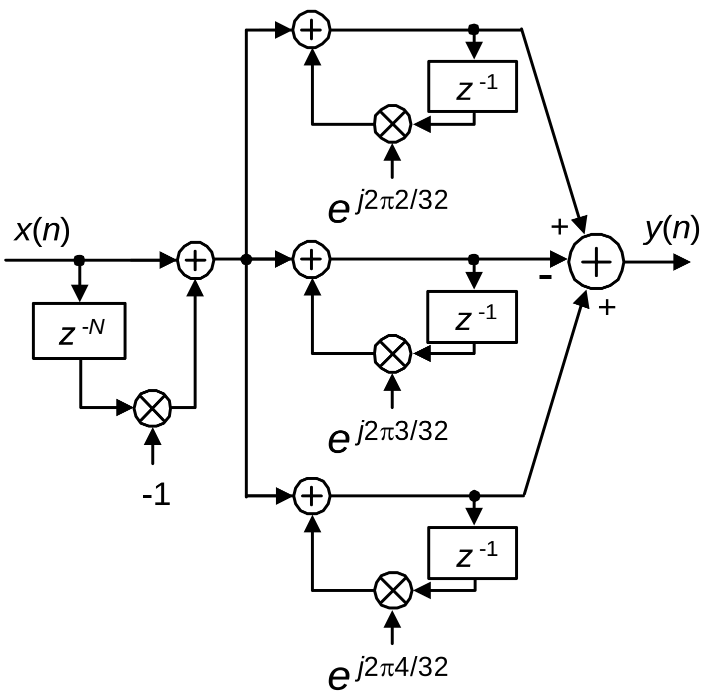
- ► <u>Here's a fascinating part of FSFs</u>: to achieve phase linearity,
  - we use $|H(k)|$'s equal to alternating + & - ones, $(-1)^k$.
  - Yielding a transfer function of

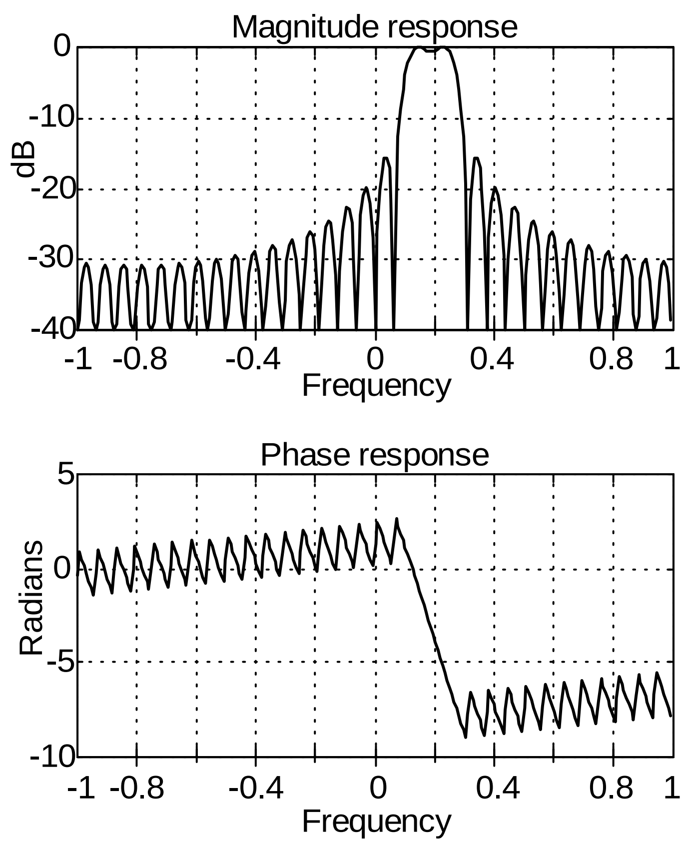
$$H_{\text{cplx,lp}}(z) = (1 - z^{-N}) \sum_{k=0}^{N-1} \frac{(-1)^k}{1 - e^{j2\pi k/N} z^{-1}}.$$

('lp' subscript means "linear-phase")

► **The (-1)^k factors simplify our FSF implementation.**

- **Complex *H*(*k*) multiplies are replaced by simple adds and subtracts.**





- **A linear-phase, even-N, complex FSF frequency response is:**
  - **the sum of individual resonators' sin(x)/x-like frequency responses, and**
  - **expressed with the wicked-looking equation:**

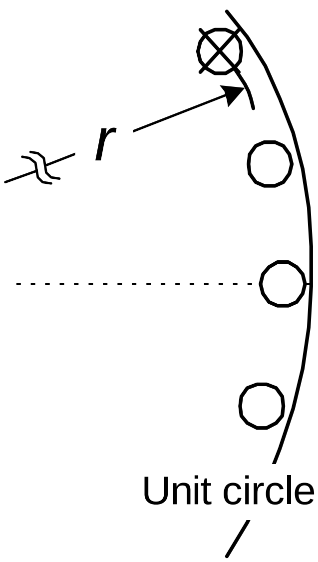
$$H_{\mathrm{cplx,lp}}(e^{j\omega}) = e^{-j\omega(N-1)/2}\sin(\omega N/2)\left[\frac{|H(N/2)|e^{-j\pi/2}}{\sin(\omega/2-\pi/2)} + \sum_{k=0}^{(N/2)-1}\frac{(-1)^k}{\sin(\omega/2-\pi k/N)} - \sum_{k=(N/2)+1}^{N-1}\frac{(-1)^k}{\sin(\omega/2-\pi k/N)}\right].$$

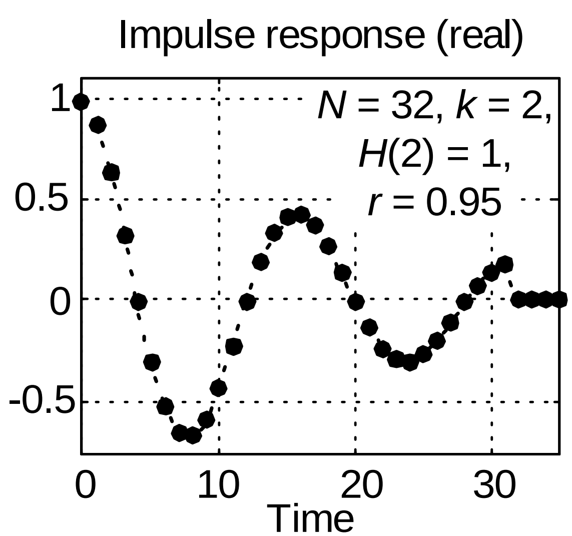
- **This expression is not as complicated as it looks.**
  - **First term inside the brackets represents the resonator centered at $k = N/2$ ($f_s/2$).**
  - **The first summation are positive-frequency resonators.**
  - **Second summation are the negative-frequency resonators.**

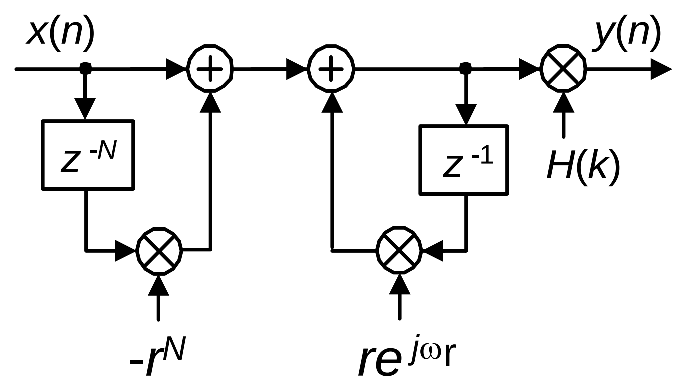
- **Reference [1] provides the derivation of the $H_{\mathrm{cplx,lp}}(e^{j\omega})$ expression.**

Ensuring FSF Stability

- **Poles and zeros are like real estate: Location, location, location!**
- **We can ensure filter stability by moving filter's zeros and poles just inside the unit circle,**
  - **at a radius of r, where the damping factor *r* is just slightly less than 1.**
- **For example, when *N* = 32, *k* = 2, *H*(2) = 2, and *r* = 0.95, we have**







► **The transfer function of a guaranteed-stable <u>single-section</u> complex FSF is**

$$H_{gs,ss}(z) = H_{comb,r<1}(z)\,H_{res,r<1}(z)\;H(k) = (1 - r^N z^{-N})\,\frac{H(k)}{1 - [re^{j2\pi k/N}]z^{-1}}\;.$$

- **Subscript 'gs,ss' means a guaranteed-stable single-section FSF.**

► **The *z*-domain transfer function of a guaranteed-stable <u>*N*-section</u> complex FSF is**

$$H_{gs,cplx}(z) = (1 - r^N z^{-N}) \sum_{k=0}^{N-1} \frac{H(k)}{1 - [re^{j2\pi k/N}]z^{-1}}$$

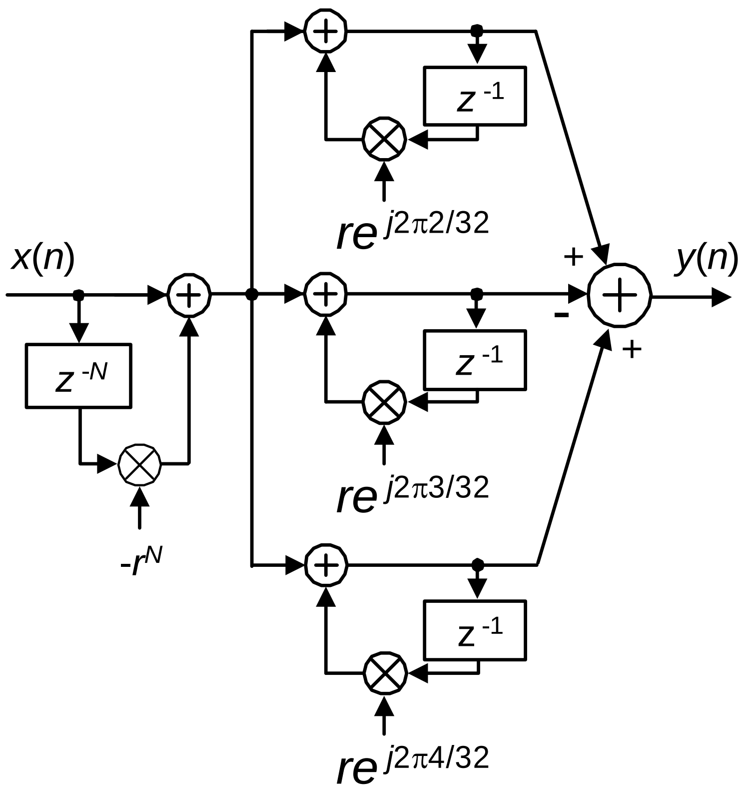
- **Subscript 'gs,cplx' means a guaranteed-stable complex multisection FSF.**

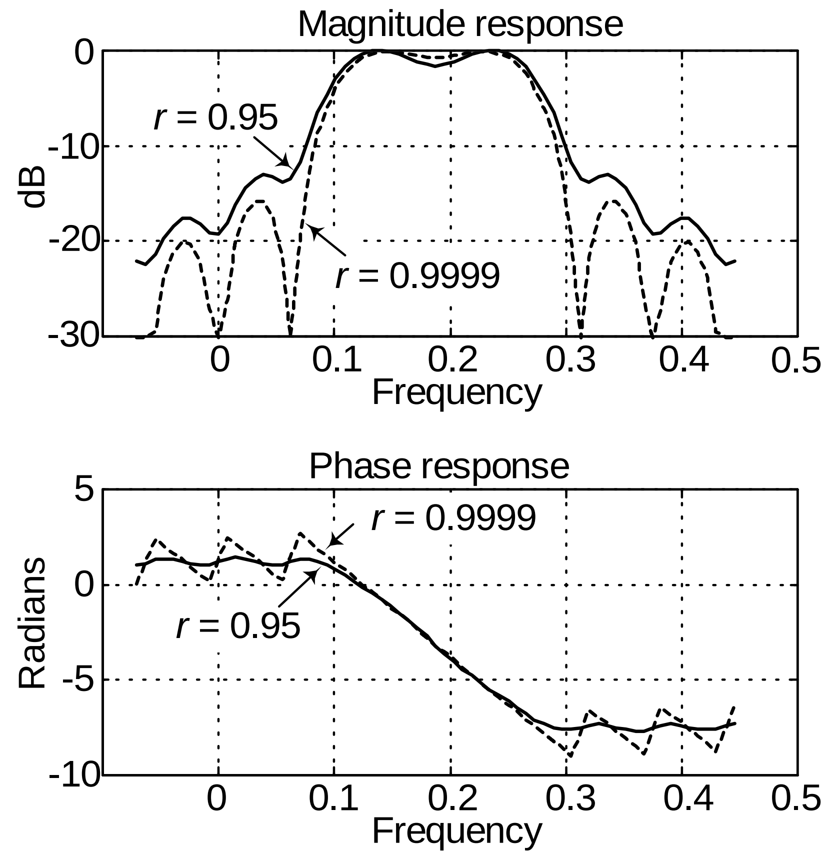
► **The frequency response of $H_{gs,cplx}(z)$ is the *messy***

$$H_{gs,cplx}(e^{j\omega}) = \sqrt{r^{N-1}}\; e^{-j\omega(N-1)/2} \sum_{k=0}^{N-1} \frac{H(k)\, e^{-j\pi k/N} \sinh[N\ln(r)/2 - jN\omega/2]}{\sinh[\ln(r)/2 - j(\omega - 2\pi k/N)/2]}.$$

- ► **Effects of zeros and poles inside the unit circle are shown below**

  - **for the two cases where *r* = 0.95 and *r* = 0.9999.**





- ► **Define *r* to be as close to unity as your binary number format allows. If integer arithmetic is used, set $r = 1\text{-}1/2^B$ where *B* is the number of bits used to represent a filter coefficient magnitude.**

Multisection Real-Valued FSFs

- **Obtain real-FSF structures (real-valued coefficients) by forcing conjugate poles,**
  - **by ensuring all non-zero $H(k)$ gain factors satisfy $H(N\text{-}k) = H^*(k)$.**

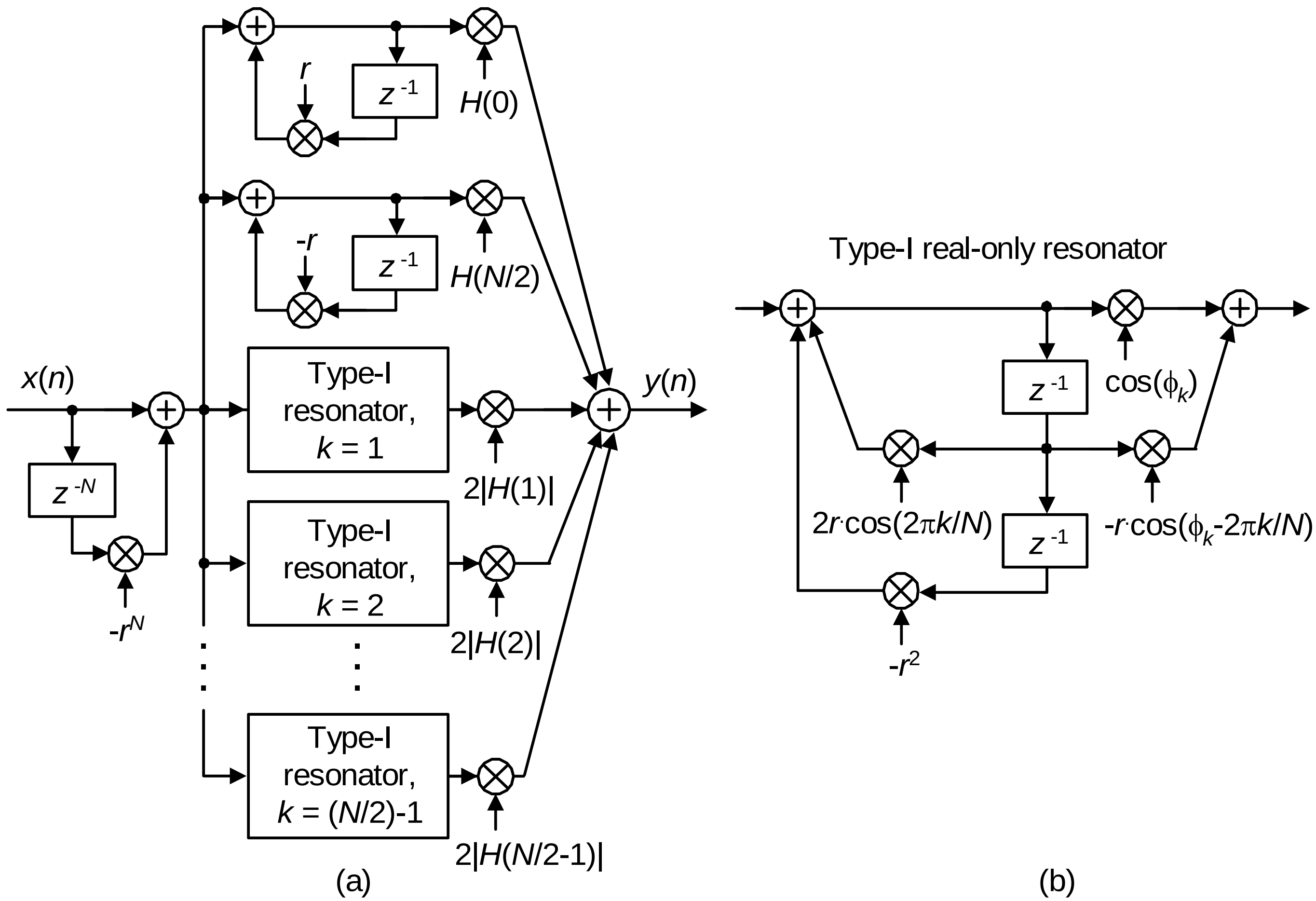
- **Conjugate pole pairs located at angles of $\pm 2\pi k/N$ radians.**

- **Transfer function of an even-N real-valued FSF is**

$$H_{\mathrm{gs,real}}(z) = H_{\mathrm{Type\text{-}I}}(z) = (1-r^N z^{-N})\left[\frac{H(0)}{1-rz^{-1}} + \frac{H(N/2)}{1+rz^{-1}} + \sum_{k=1}^{N/2-1}\frac{2|H(k)|\,[\cos(\phi_k)-r\cos(\phi_k-2\pi k/N)z^{-1}]}{1-[2r\cos(2\pi k/N)]z^{-1}+r^2z^{-2}}\right]$$

  - **Subscript 'gs,real' means a guaranteed-stable real-valued multisection FSF**
  - **ϕ_k is the desired phase angle of the kth section.**

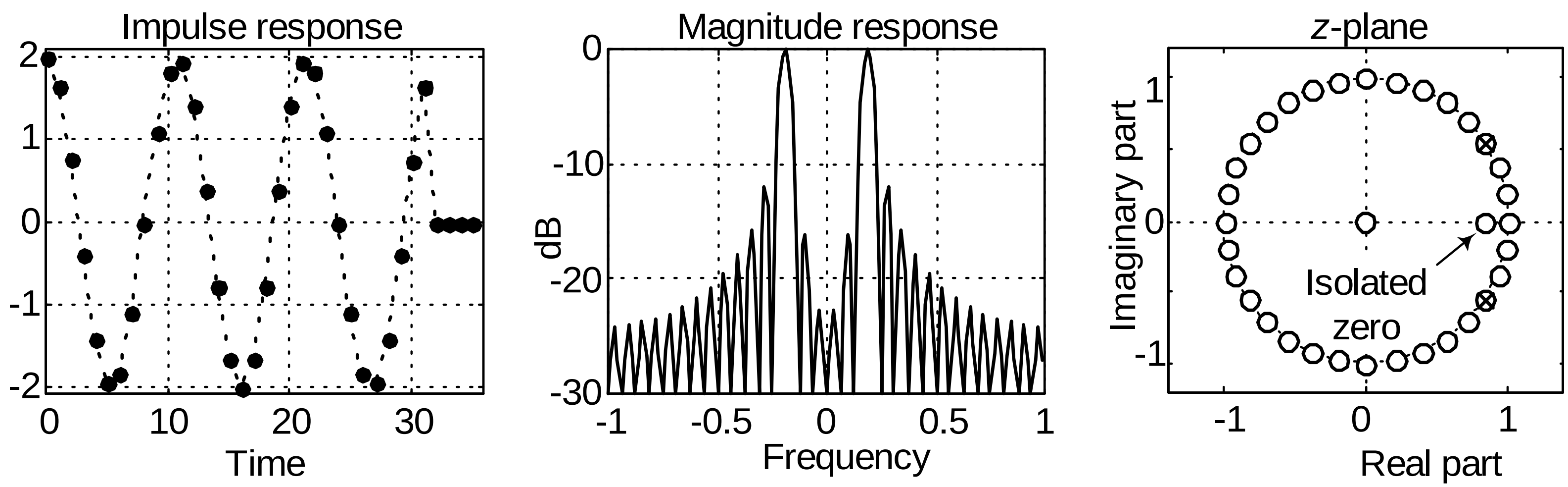
► We call this real-valued FSF a *Type-I real FSF* whose implementation is:



► For lowpass FSFs the stage associated with the $H(N/2)$ gain factor would not be implemented.

► For bandpass FSFs neither stage associated with the $H(0)$ and $H(N/2)$ gain factors would be implemented.

► The behavior of a single-section Type-I real FSF with N = 32, k = 3, H(3) = 1, r = 0.99999, and $\phi_3 = 0$ is :

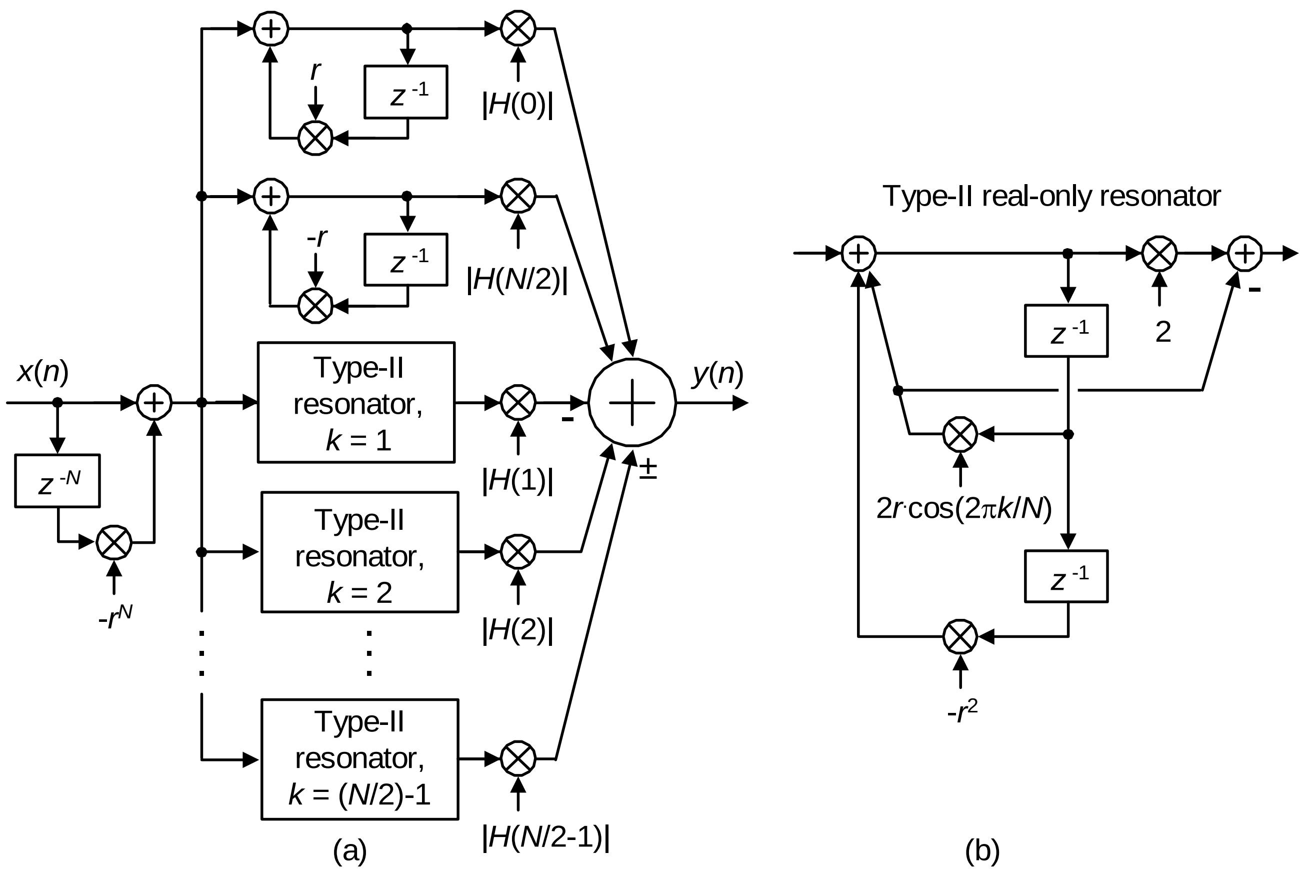


(a) (b) (c)

► An alternate version of the Type-I FSF, with a simplified resonator structure, can be developed:

- Set all ϕ_k values equal to zero,
- move the gain factor of 2 inside the resonators, and
- incorporate the alternating signs in the final summation to linearize phase.

► This gives us a ***Type-II real FSF*** whose implementation is:



► ***Type-II*** **resonators require fewer multiplies than do** ***Type-I*** **resonators.**

- The transfer function of this Type-II real FSF is

$$H_{\text{Type-II}}(z) = (1-r^N z^{-N})\left[\frac{|H(0)|}{1-rz^{-1}} + \frac{|H(N/2)|}{1+rz^{-1}} + \sum_{k=1}^{N/2-1} \frac{(-1)^k\,|H(k)|[2-2r\cos(2\pi k/N)z^{-1}]}{1-[2r\cos(2\pi k/N)]z^{-1}+r^2 z^{-2}}\right].$$

- Neither the Type-I or the Type-II FSF have exactly linear phase.

- They have isolated zeros located at $z = r\cos(2\pi k/N)$, when $\phi_k = 0$,
  - with no accompanying reciprocal zeros located outside the unit circle at $z = 1/[r\cos(2\pi k/N)]$,
  - This causes phase nonlinearity.
  - Their group delays can have a peak-to-peak fluctuation of up to two sample periods ($2/f_s$).

- Type-I and -II FSFs are the most common types described in the literature of FSFs.
  - Their nonlinear phase has not received sufficient attention or analysis.

- Let's take steps to obtain linear phase by repositioning the isolated zero.

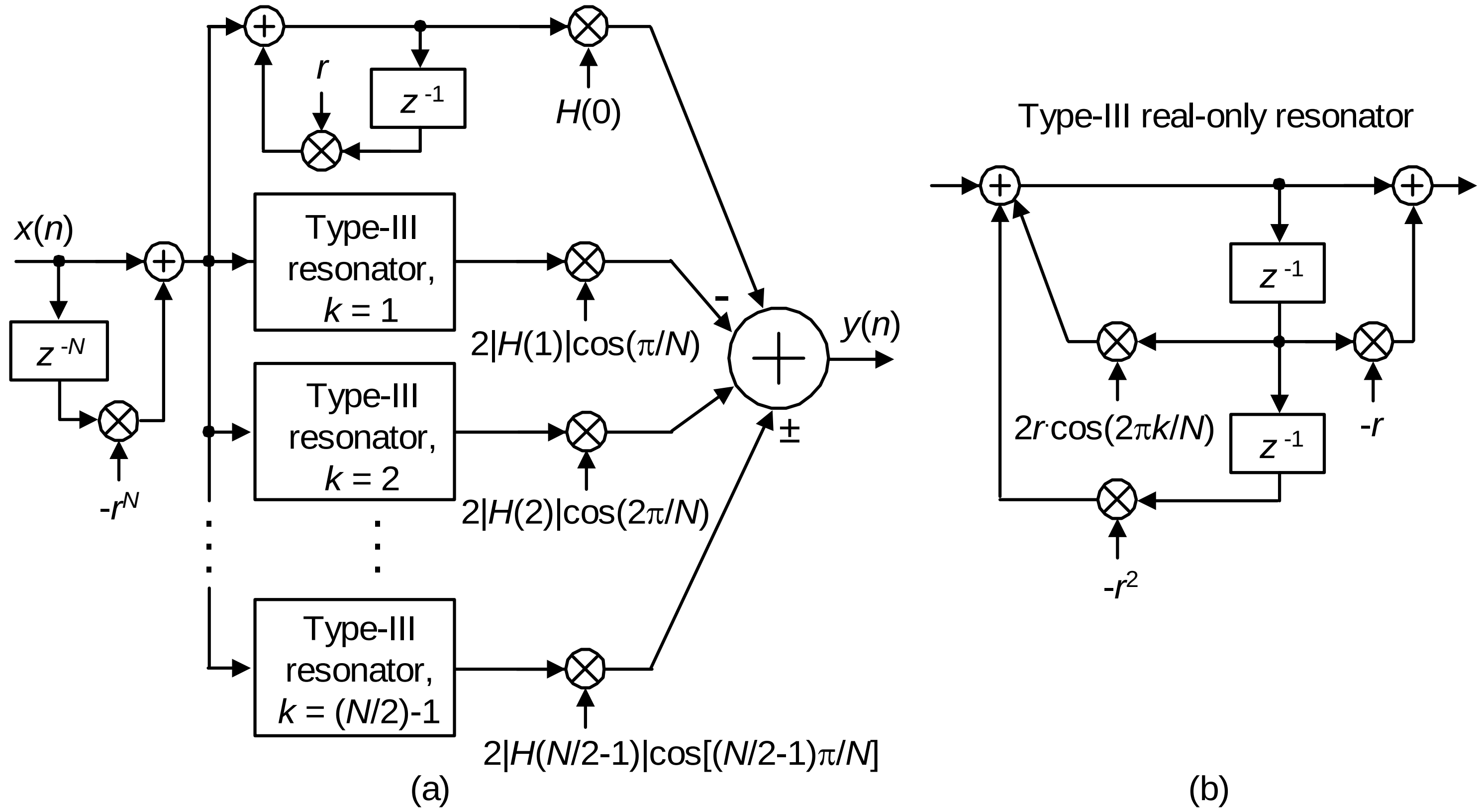
Linear-phase Multisection Real-Valued FSFs

- **We can achieve exact real-FSF phase linearity by:**
  - **by setting $\phi_k = \pi k/N$.**
  - **This moves the Type-I FSF's isolated zero on top of the comb filter's zero located at $z = r$,**

- **The transfer function of this linear-phase *Type-III real FSF* is**

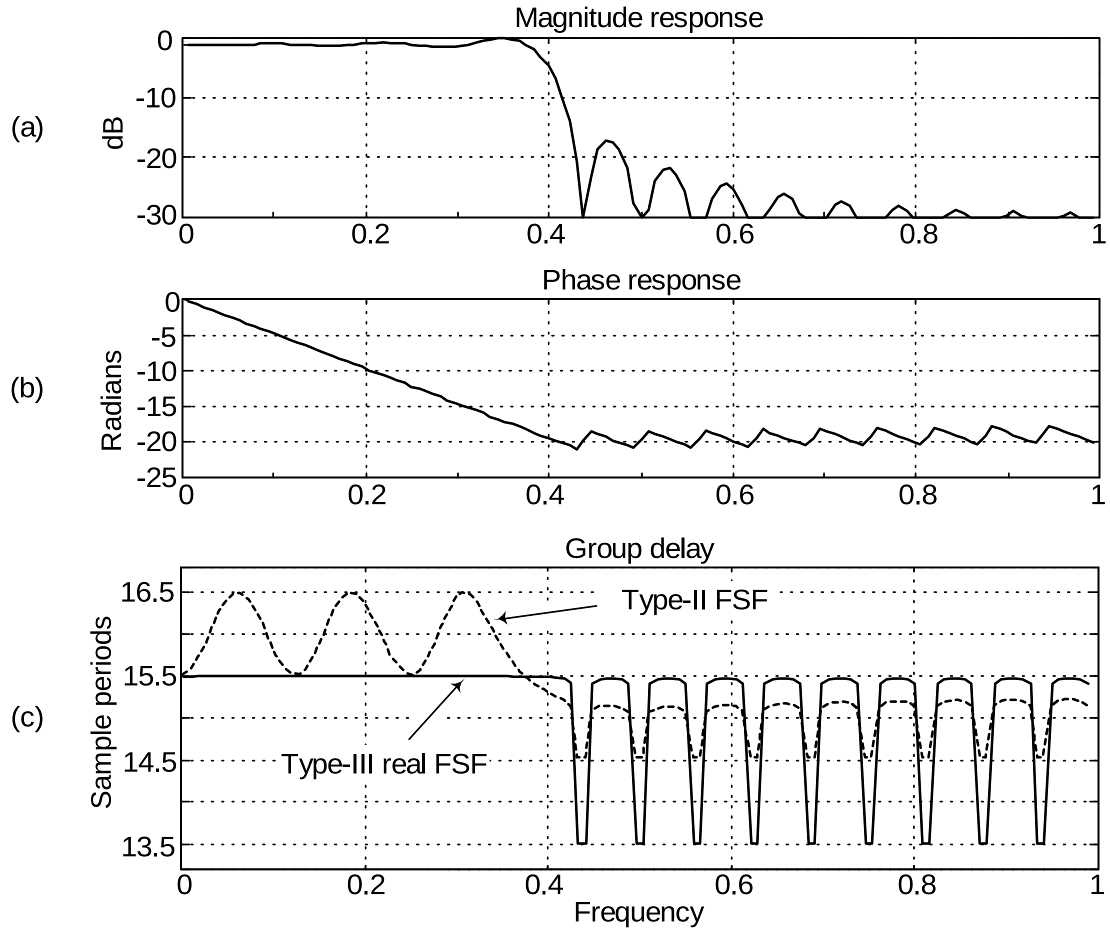
$$H_{\text{Type-III}}(z) = (1 - r^N z^{-N})\left[\frac{|H(0)|}{1 - rz^{-1}} + \sum_{k=1}^{N/2-1} \frac{2(-1)^k |H(k)|\cos(\pi k/N)[1 - rz^{-1}]}{1 - [2r\cos(2\pi k/N)]z^{-1} + r^2 z^{-2}}\right].$$

► **The implementation of the linear-phase Type-III real FSF is**



► **Here's the frequency-domain performance of an eight-section Type-III FSF,**

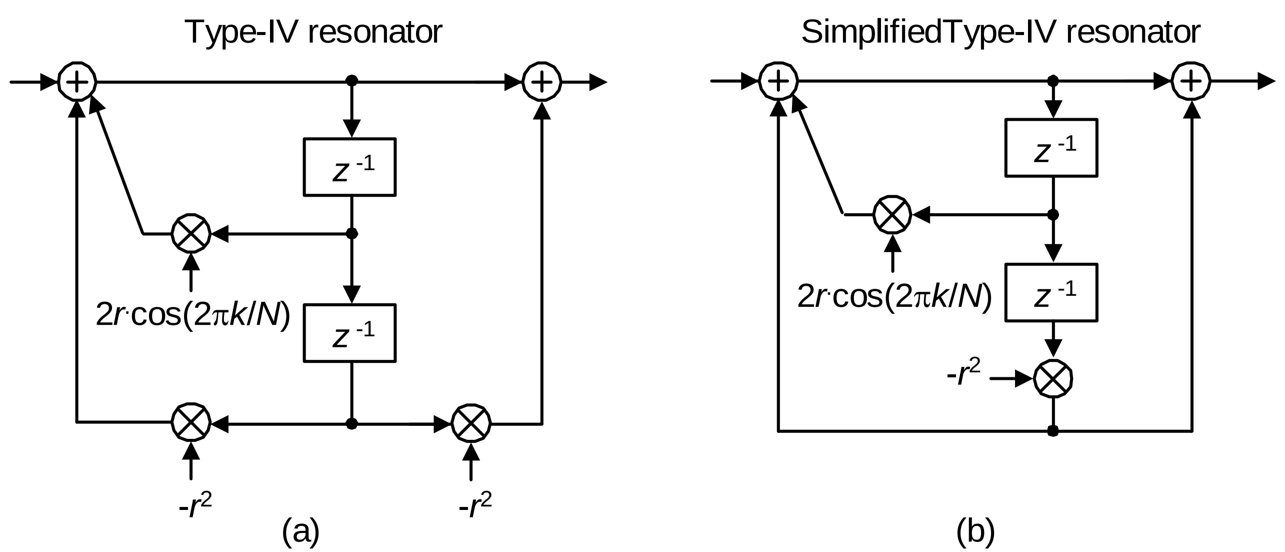
- **for N = 32 where the eight sections begin at DC ($0 \le k \le 7$).**
- **Notice the improved Type-III phase linearity, a constant group delay of (N-1)/2 samples.**



An Proposed Real-valued FSF

► **There are many real-valued resonators that can be used in FSFs.**

► **Of particular interest is the <u>proposed</u> *Type-IV resonator*, shown as:**



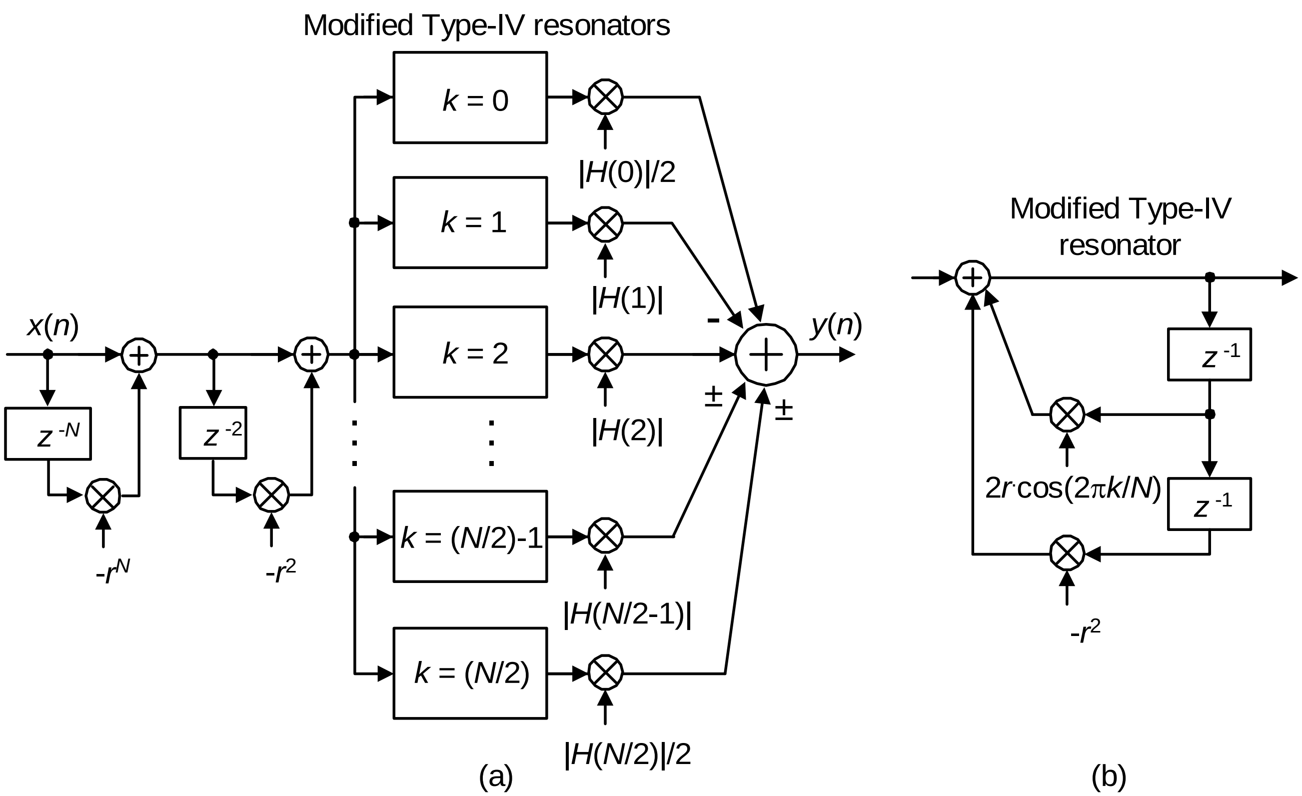
► **This resonator deserves our attention because it:**

- **is guaranteed-stable, exhibits highly linear phase,**
- **is computationally efficient, can implement highpass FIR filters, and**
- **yields better stopband attenuation performance than Type-I, -II, & -III FSFs.**

- **The Type-IV FSF transfer function is**

$$H_{\text{Type-IV}}(z) = (1 - r^N z^{-N}) \sum_{k=0}^{N/2} \frac{(-1)^k |H(k)| (1 - r^2 z^{-2})}{1 - 2r\cos(2\pi k/N)z^{-1} + r^2 z^{-2}}$$

- **We reduce the number of multiply and addition operations by**
  - **by implementing the $(1-r^2z^{-2})$ term in the numerator as a second-order comb filter as shown below.**



► **The '±' symbols in remind us when *N* is even, *k* = *N*/2 could be odd or even.**

► **The Type-IV FSF frequency response is**

$$H_{\text{Type-IV}}(e^{j\omega}) = e^{-j\omega N/2} \sum_{k=0}^{N/2} \frac{(-1)^k |H(k)|[\cos(\omega N/2 - \omega) - \cos(\omega N/2 + \omega)]}{\cos(2\pi k/N) - \cos(\omega)}.$$

► **The even-*N* peak resonance magnitude response for a single Type-IV FSF section is**

$$|H_{\text{Type-IV}}(e^{j\omega})|_{\omega=2\pi k/N} = N, \quad k = 1, 2, ..., \frac{N}{2} -1$$

- and resonant magnitude gains at *k* = 0 (DC), and *k* = *N*/2 (f_s/2), are

$$|H_{\text{Type-IV}}(e^{j\omega})|_{\omega=0} = |H_{\text{Type-IV}}(e^{j\omega})|_{\omega=\pi} = 2N.$$

► **We've covered a lot of ground so far.**

► **Here's a summary of real-valued FSFs.**

Summary of even-*N* real FSF properties

Real FSF type	Group delay	Multiplies	Adds	Remarks
Type-I	$\frac{N}{2}$	5	3	Real-coefficient FSF. Phase is only moderately linear.
Type-II	$\frac{N}{2}$	3	3	Modified, and more efficient, version of the Type-I FSF. Phase is only moderately linear. A binary left shift may eliminate one resonator multiply.
Type-III	$\frac{N-1}{2}$	4	3	Very linear phase. Cannot implement a linear-phase highpass filter.
Type-IV	$\frac{N}{2}$	2	2	Very linear phase. Improved stopband attenuation. Usable for lowpass, bandpass, or highpass filtering. Lean mean filtering machine.

► **The section gain at their resonant frequencies is *N* for all four real-valued FSFs.**

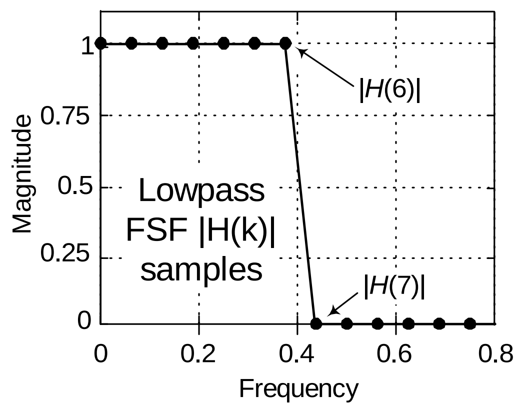
Improving Performance with Transition Band Coefficients

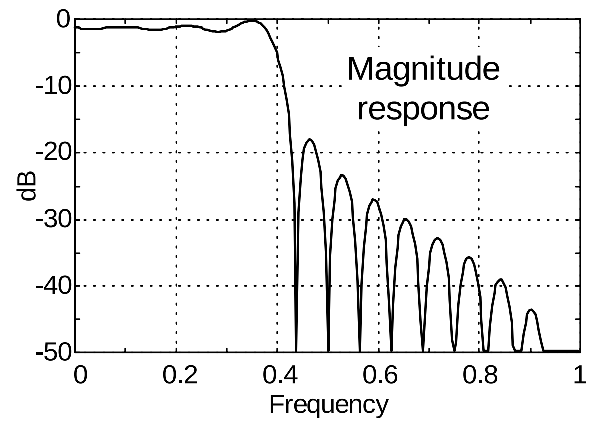
► **We can increase FSF stopband attenuation defining non-unity |*H*(*k*)| magnitude samples**

- **in the transition region, between the passband and stopband.**

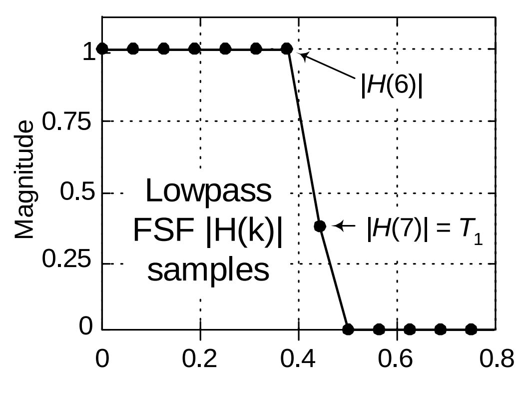
► **Consider a seven-section lowpass Type-IV FSF with *N* = 32.**

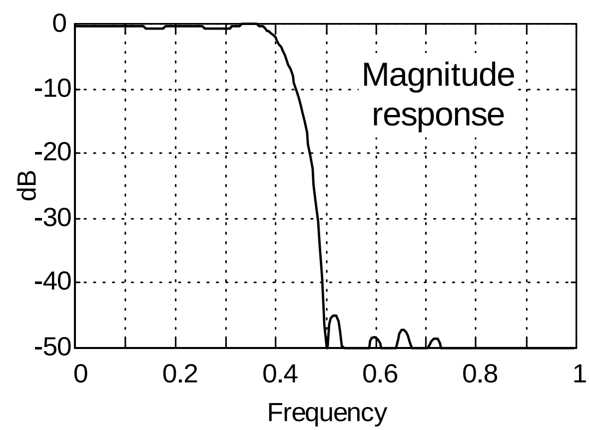
No transition coefficient used.





Single transition coefficient used.



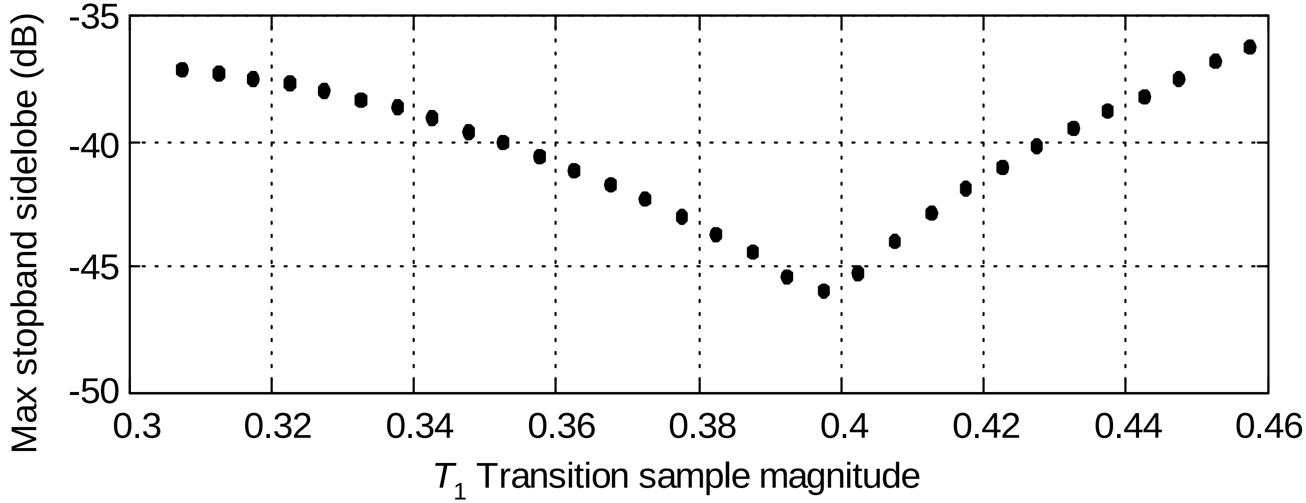


► **Setting coefficient T_1 = 0.389 reduces the abruptness in the transition region.**

- **Results in reduced passband ripple and the improved stopband attenuation.**
- **Has computational cost of an additional FSF section and increased width of the transition region.**

► **Assigning a coefficient value of T_1 = 0.389 was not arbitrary nor magic.**

► **For the seven-section Type-IV FSF an optimum value for T_1 exists, as shown below.**



- **Further stopband attenuation is possible if two transition coefficients, T_1 and T_2, are used,**
  - **such that $0 \le T_2 \le T_1 \le 1$.**
  - **Each additional transition coefficient improves the stopband attenuation by roughly 25 dB.**

- **Finding the optimum values for the *T* coefficients is a daunting task.**

- **Optimum transition region coefficient values depend on:**
  - **the number of unity-gain FSF sections, the value of *N*, and**
  - **the number of transition coefficients used.**

- **There's no closed form equation available to calculate optimum transition coefficient values. ☹**

- **We must search for them empirically, but**
  - **commercial mathematical software packages have built-in *optimization* functions to facilitate this computationally-intensive search.**

- **Happily, for the Type-IV FSF, <u>tables of optimum transition coefficients are available.</u>**
  - **<u>Compiled by the author and are provided in Reference [1].</u>** ☺

- **The FSF design tables in Ref [1] look something like:**

$N = 16$

BW	Atten	T1		BW	Atten	T1	T2
1	-44.9	0.41924081		1	-76.5	0.56626687	0.07922718
2	-45.8	0.38969818		2	-77.2	0.55487263	0.08012238
3	-47.3	0.36942214		3	-81.2	0.53095099	0.07087993
4	-49.6	0.34918551		4	-87.7	0.49927622	0.05813368

$N = 24$

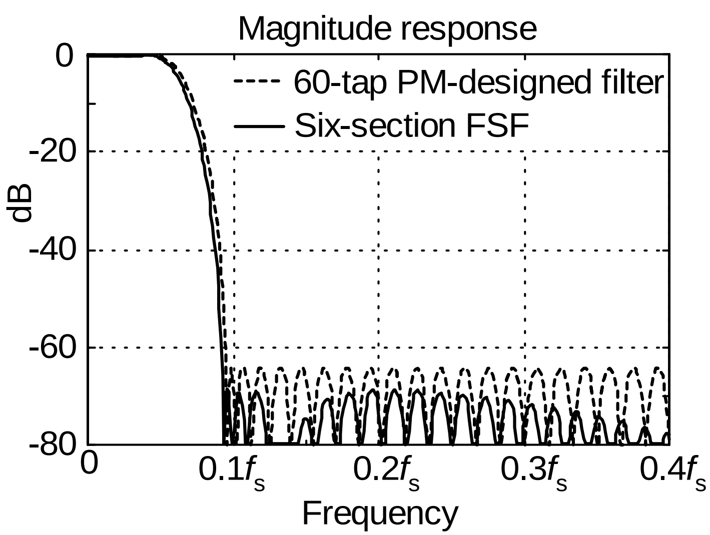
BW	Atten	T1		BW	Atten	T1	T2
1	-44	0.42452816		1	-73.6	0.57734042	0.08641861
2	-44.1	0.40042889		2	-72.5	0.57708274	0.09305238
3	-44.9	0.38622106		3	-72.9	0.56983709	0.09177956
4	-45.7	0.37556064		4	-73.8	0.55958351	0.08770698
5	-46.5	0.36663149		5	-75.6	0.54689579	0.08202772

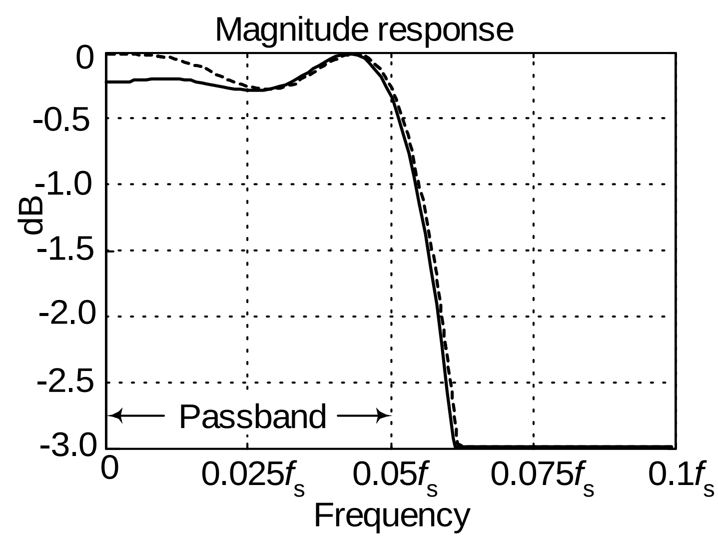
Type-IV FSF Example

► **Consider the design of a linear-phase lowpass FIR filter with:**

- **cutoff frequency of 0.05f_s, stopband begins at 0.095f_s,**
- **max passband ripple of 0.3 dB (p-p), minimum stopband attenuation of 65 dB.**

► **A six-section Type-IV FSF with N = 62 and r = 0.99999, its frequency-domain performance satisfies our requirements and is the solid curve below.**





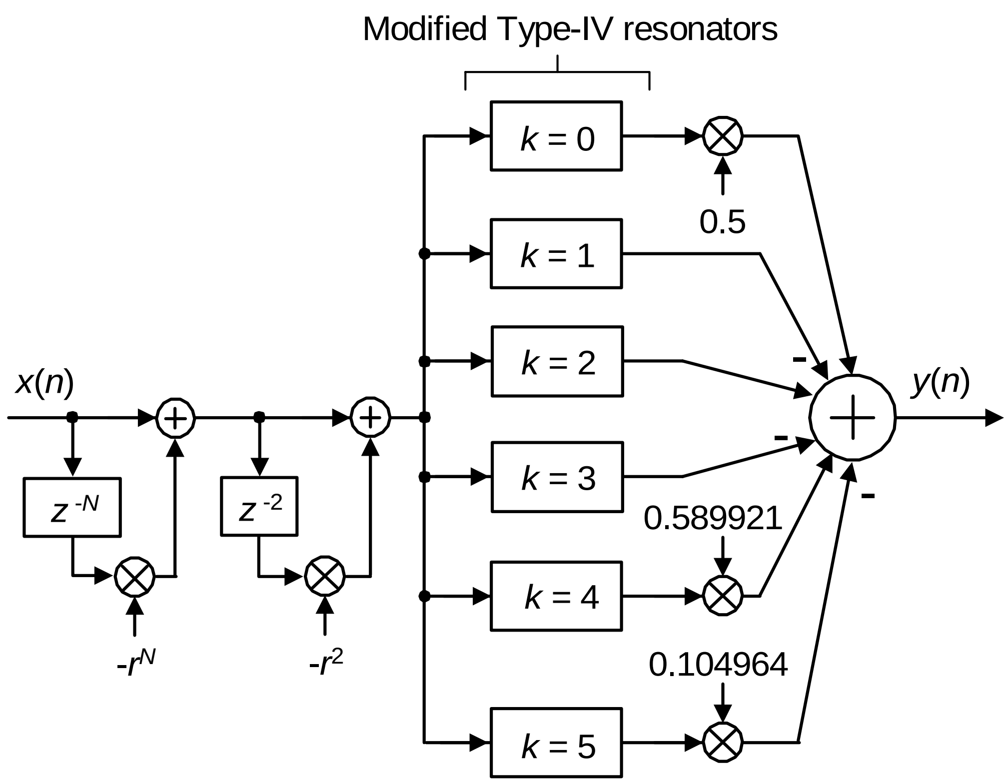
Two FSF transition region coefficients are:

- **$|H(4)| = T_1 = 0.589921$, and $|H(5)| = T_2 = 0.104964$.**

► **A PM-designed filter implemented using a *folded* nonrecursive FIR structure**

- **requires 30 multiplies and 59 adds per output sample.**

► **The Type-IV FSF requires <u>only</u> 17 multiplies and 19 adds per output sample.** ☺



When to Use an FSF

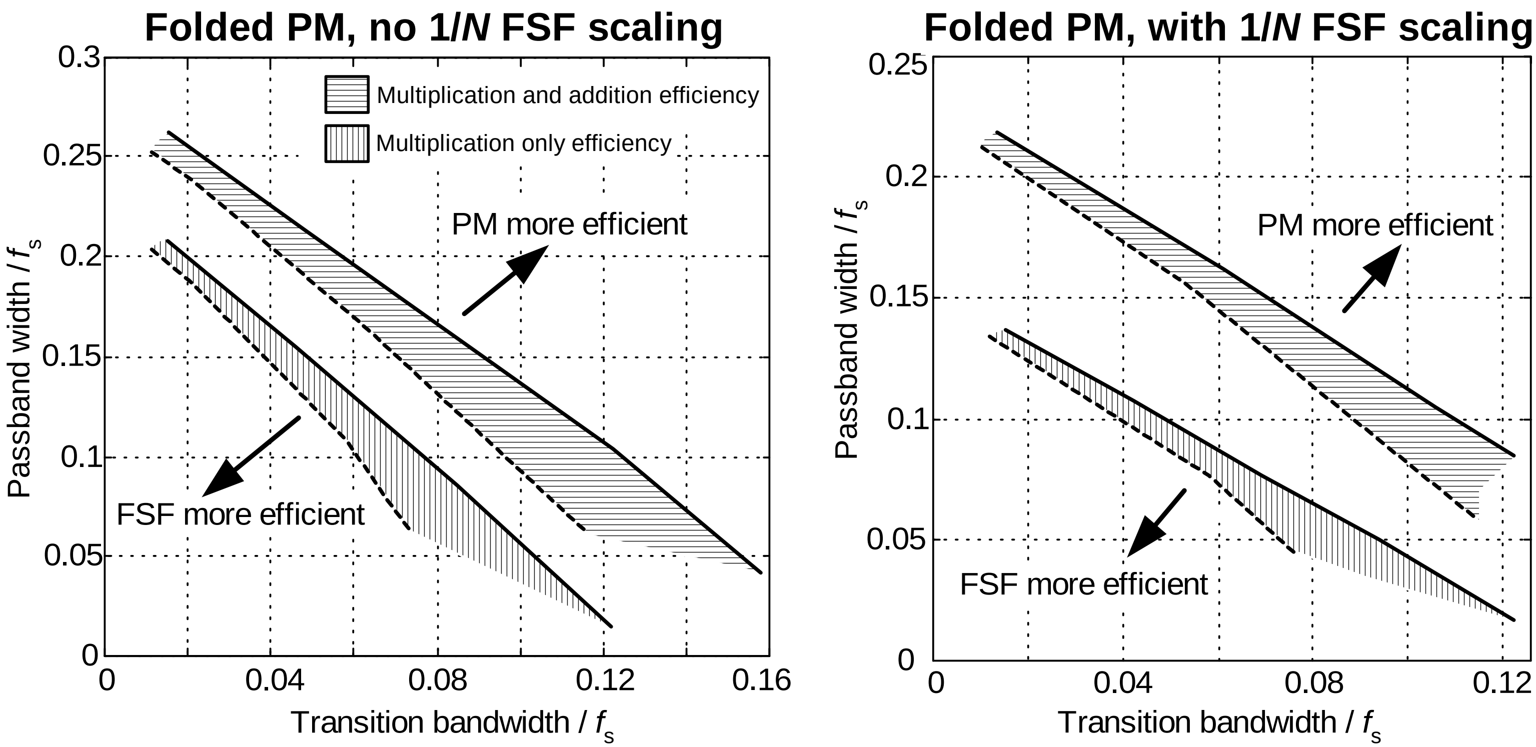
- **In this section <u>we answer the burning question</u>;**
  - **when should you use a Type-IV FSF instead of a PM-designed FIR filter?**

- **The answer depends on:**
  - **the PM FIR filter implementation (folded, or non-folded),**
  - **do multiplications and additions require an equal number of clock cycles, and**
  - **whether FSF output scaling, by 1/*N*, is needed.**

- **We have four scenarios to consider:**
  1) **Folded-structure FIR & no 1/*N* FSF scaling,**
  2) **Folded-structure FIR & 1/*N* FSF scaling used,**
  3) **Non-folded FIR & no 1/*N* FSF scaling, and**
  4) **Non-folded FIR & 1/*N* FSF scaling used.**

► **Computational comparison between even-*N* Type-IV FSFs and *folded* PM-designed filters.**

- **The bands represent desired filter performance parameters where a Type-IV FSF and a *folded* PM designed filter have roughly equal computational workloads.**



► **<u>If the desired FIR filter transition region width and passband width combination (a point) lies beneath the *comparison band of interest*, then a Type-IV FSF is more computationally efficient than a PM-designed filter.</u>**

► **The solid lines are for three transition coefficients, and the dotted lines are for one transition coefficient.**

► **For filter implementations where multiplies and adds require equal processor clock cycles,**

- **the ▤ band is the *comparison band of interest*.**

► **For filter implementations where a single multiply requires many more processor clock cycles than a single addition,**

- **the ▥ band is the *comparison band of interest*.**

► **The comparison charts assume an FSF damping factor of r = 0.99999.**

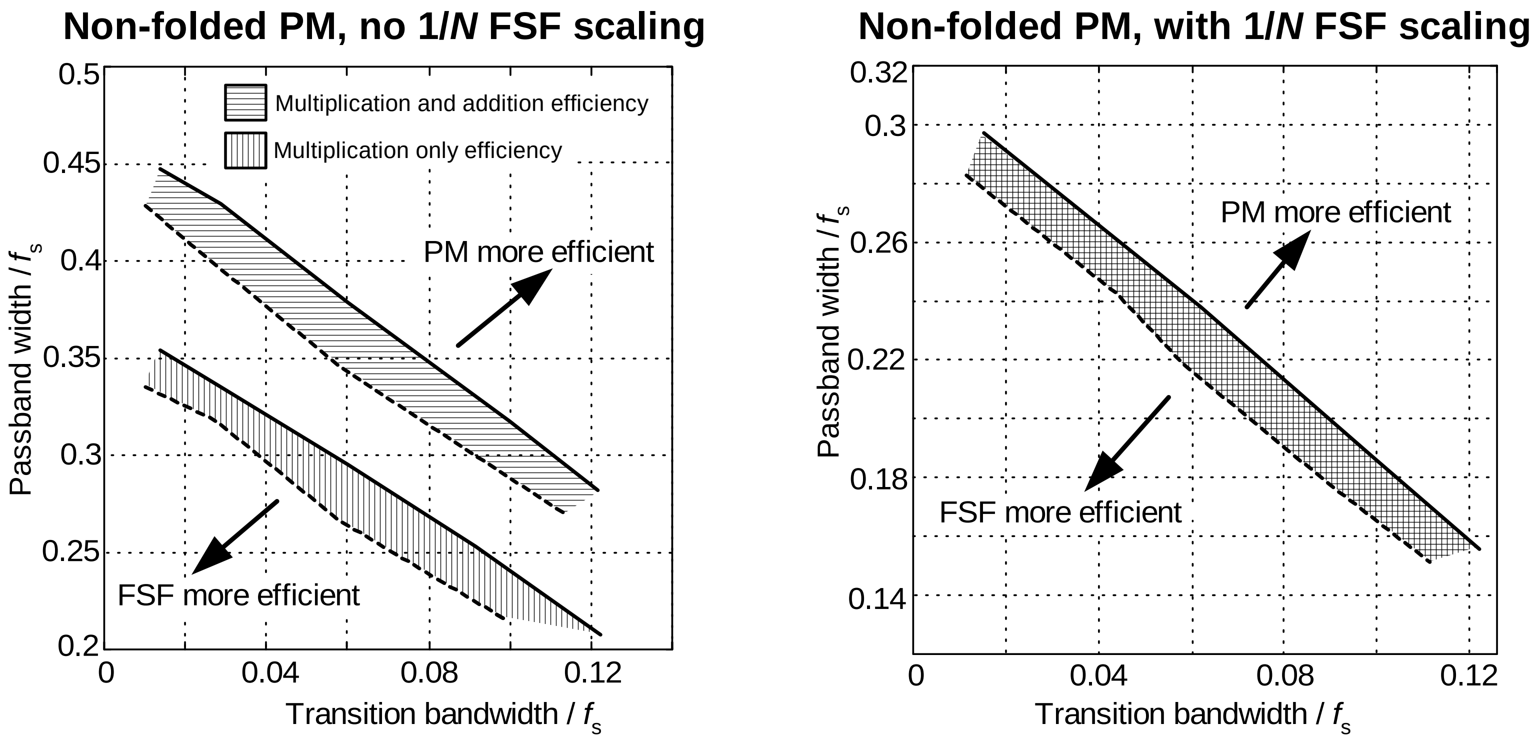
► **Performance criterion levied on the PM filter are typical Type-IV FSF properties (when floating-point coefficients are used) given in the following table.**

Parameter	1-coefficient	2-coefficient	3-coefficient
Passband peak-peak ripple (dB)	0.7	0.35	0.16
Minimum stopband attenuation (dB)	-45	-68	-95

► **FSFs have a gain of N; in fixed point implementations the gain may cause overflow errors.**

- **A gain reduction by a scaling factor of 1/N may be necessary (at resonators' outputs).**
- **In this case, refer to the bands in the right panel of the previous page.**

► **Computational comparison between even-*N* Type-IV FSFs and *non-folded* PM-designed filters.**



► **For a non-folded PM FIR filter, the number of multiplications & additions are roughly equal.**

► **When 1/*N* scaling is used for the FSF, the number of multiplications & additions are equal.**

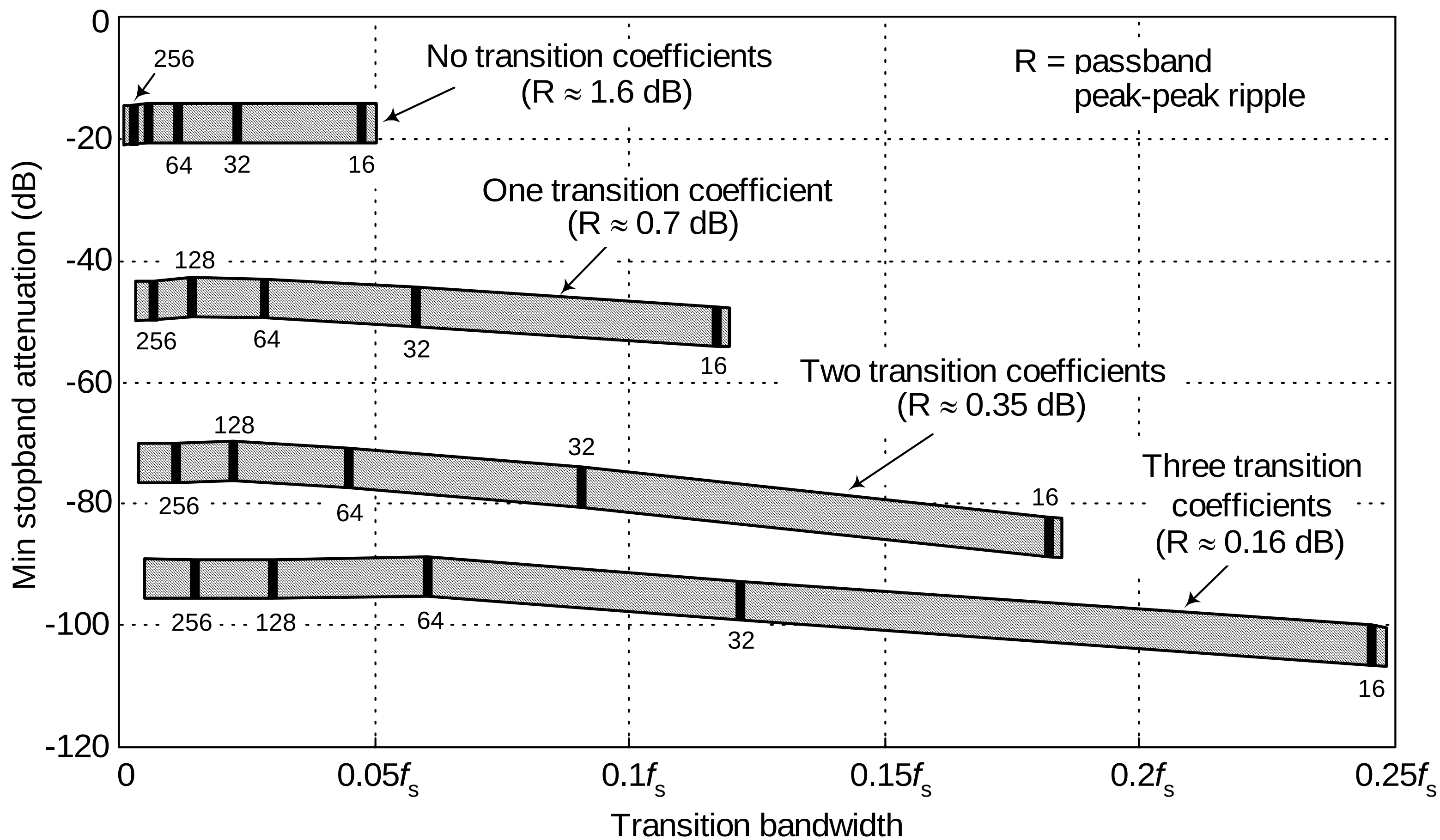
► **These conditions explain why the right panel in the above figure has only one band;**

- **for both filters, the number of *multiplications and additions* is roughly twice the number of *multiplication- only*.**

► **To conclude our *When To Use an FSF Filter* discussion: <u>an FSF should be considered</u> if:**

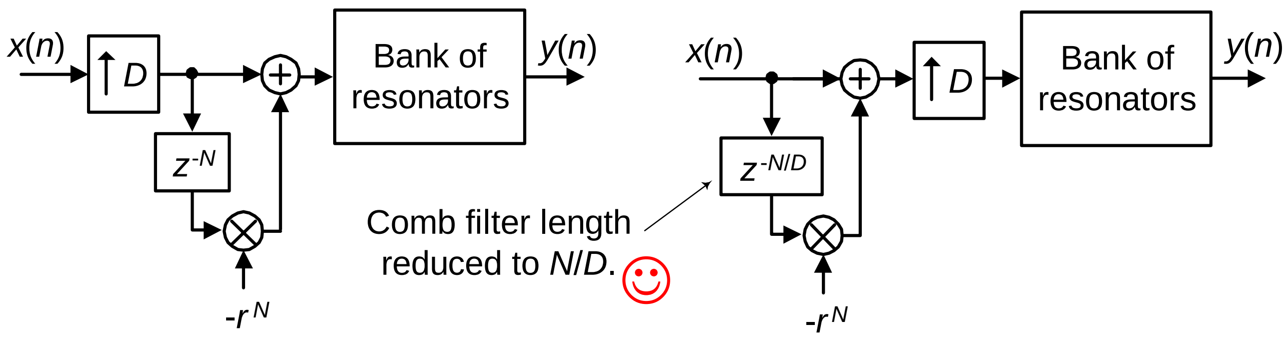
- **your desired filter's passband & transition widths are below the implementation *comparison bands of interest* in the above figures, and**

- **your desired filter's stopband attenuation is within, or below, one of the bands in the following figure.**

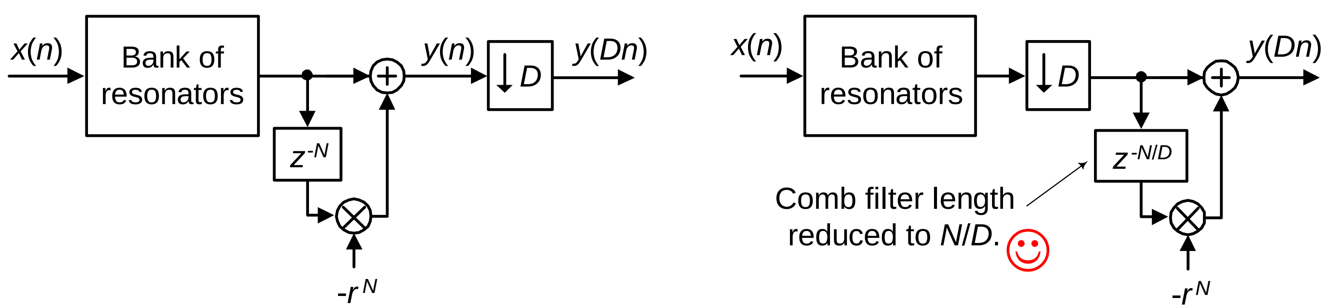


FSFs and With Sample Rate Conversion (SRC)

- **FSFs need *N* memory locations to implement a length-*N* comb filter.**
- **For interpolation, comb filter storage can be reduced,**
  - **swap order of the "*upsample by D*" and the comb filter.**



- **Likewise for decimation, comb filter storage can be reduced,**
  - **swap order of the "*downsample by D*" and the comb filter.**



- **Of course *D* is chosen such that the ratio *N*/*D* is an integer.**

FSF Summary

- **We've introduced the structure and performance of FSFs.**

- **Emphasis given to the practical issues**
  - **phase linearity,**
  - **stability, and**
  - **computational workload.**

- **FSFs are more efficient, for certain applications, than PM-designed FIR filters.**
- **FSFs are modular; their sections are computationally identical and well understood.**
- **Although recursive, FSFs can be designed to be stable and linear phase.**
- **Design tables of optimum transition region coefficients are available.**

► **More FSF details,**

- **math derivations,**
- **transition region coefficients design tables,**
- **design examples,**
- **design guidelines, and**
- **additional literature references are provided in:**

Reference [1]:

***Understanding Digital Signal Processing*, 2nd Ed., by R. Lyons, Prentice Hall, Upper Saddle River, New Jersey, 2004**